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On Λ-Fractional Analysis and Dirac Delta Functions

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Abstract

Λ-fractional analysis is extended, just to include fractional derivatives with fractional orders higher than one. It is proposed that Λ-fractional derivatives of orders higher than one are directly related to the Λ-fractional derivatives of orders less than one. The proposed Λ-fractional derivatives are applied to the Dirac delta functions.

Keywords: Fractional Λ-derivative; fractional Λ-space; non-local character; γ-fractional order; Dirac Delta function; γ order-Λ-Fractional Dirac Delta function.

Introduction

Fractional calculus is a quite strong mathematical tool, especially for non-local mathematical analysis, demanded by specific problems of physics, mechanics, engineering, etc. Eringen [1] has proposed non-local extensions of two fundamental laws in physics. a) The energy balance law to remain in global form and b) a material point is considered to be attracted by all points of the body, at all past times. Leibnitz [2] proposed the fractional derivative in 1695, and many famous researchers [3-6] worked on the topic presenting various types of fractional calculus. There exists extensive literature concerning applications of fractional calculus in viscoelasticity, Rabotnov [7], Bagley & Torvik [8,9], Atanackovic [10], Mainardi [11]. Nevertheless, all the well-known fractional derivatives, do not access all the properties of the derivatives, because they fail to satisfy Differential Topology prerequisites for corresponding to differentials, generating Differential Geometry. Those well-known rules are Chillingworth [12]:

1. Linearity D(af(x)+bg(x))=aDf(x)+bDg(x) (1.1)
2. Leibniz rule D(f(x)·g(x))=Df(x)·g(x)+f(x)·Dg(x) (1.2)
3. Chain rule D(g(f))(x)=Dg(f(x)·Df(x) (1.3)

Lazopoulos [14], presented a fractional derivative satisfying both the prerequisites of Differential Topology and the fractional postulates, able to generate Fractional Differential Geometry, Lazopoulos [15]. It was proposed the Λ-Fractional Analysis with the basic characteristics of the Λ-Fractional derivative and Λ-fractional space where the fractional derivatives behave as conventional local ones. That analysis has been applied in continuum mechanics, Lazopoulos [16, 17,18]. The Λ-fractional derivative (Λ-FD) is defined for $0<\gamma<1$ as:

$$ {}_{a}^{\Lambda}D_{x}^{\gamma} f(x) = \frac{{}_{\alpha}^{RL}D_{x}^{\gamma} f(x)}{{}_{a}^{RL}D_{x}^{\gamma} x} \tag{1.4}$$

Recalling the definition of the Riemann Liouville's fractional derivative, Eq.(3), the Λ-FD is expressed by,

$$ {}_{a}^{\Lambda}D_{x}^{\gamma} f(x) = \frac{\dfrac{d\, {}_{a}I_{x}^{1-\gamma} f(x)}{dx}}{\dfrac{d\, {}_{a}I_{x}^{1-\gamma} x}{dx}} = \frac{d\, {}_{a}I_{x}^{1-\gamma} f(x)}{d\, {}_{a}I_{x}^{1-\gamma} x} \tag{1.5}$$

Further, the Λ-fractional space is defined by (X, F(X)) with,

$$ X = {}_{a}I_{x}^{1-\gamma} x, \qquad F(X) = {}_{a}I_{x}^{1-\gamma} f(x(X)) \quad . \tag{1.6}$$

The Λ-FD exhibits all the properties of the conventional local derivatives, in the Λ-fractional space (X, F(X)). Hence, Fractional Differential Geometry may be generated as a conventional differential geometry in the fractional Λ- space, (X, F(X)). Then the results may be transferred to the initial one using the relation,

$$ f(x) = {}_{a}^{RL}D_{x}^{1-\gamma} F\big(X(x)\big) = {}_{\alpha}^{RL}D_{x}^{1-\gamma} I^{1-\gamma} f(x). \tag{1.7}$$

In case the contribution of the right side fractional derivative should be taken into consideration, the Λ-fractional space may be defined with,

$$ I^{1-\gamma} f(x) = \frac{1}{2}\big(I_{x}^{1-\gamma} f(x) + I_{b}^{1-\gamma} f(x)\big) = \frac{1}{2}\big(I_{x}^{1-\gamma} f(x) + I_{b}^{1-\gamma} f(x)\big). \tag{1.8}$$

Trying to formulate the Λ-fractional analysis into the context of the existing fractional calculus, conforming with the demands, as have been postulated by Ross [6,7], the Λ-fractional derivative for γ>1 is formulated as follows: If $k-1 \leq \gamma < k$, the Λ-fractional derivative is defined by,

$$X = {}_aI_x^{\gamma_0} x, \qquad F(X) = {}_aI_x^{\gamma_0} f(x(X)), \qquad \gamma_0 = \gamma / k \quad . \tag{1.9}$$

Then, the Λ-fractional derivative is defined by,

$${}_a^{\Lambda}D_x^{\gamma_0} f(x) = \frac{\frac{dF(x(X))}{dx}}{\frac{dX}{dx}} = \frac{dF(x(X))}{dX} \qquad 0 \leq \gamma_o < 1 \ . \tag{1.10}$$

Hence,

$${}_a^{\Lambda}D_x^{\gamma} f(x) = \frac{d^k F(x(X))}{d^k X} = \Phi(X(x)) \tag{1.11}$$

The Λ-fractional derivative coincides with the R-L fractional derivative when 1-γ=0 and consequently with the conventional derivative. The other steps remain the same with the transferring rule,

$$f(x) = {}_a^{RL}D_x^{1-\gamma_0} \Phi\big(X(x)\big), \qquad \text{with} \quad 0 < \gamma - k < 1 \tag{1.12}$$

Therefore, the Λ-fractional derivative has been extended just to include orders higher than one.

$$f(x) = {}_0^{RL}D_x^{1-\gamma_0} \Phi\big(X(x)\big) = {}_0^{RL}D_x^{1-\gamma_0} I_x^{1-\gamma_0} f(x).$$

Example

For a better explanation of the initial and the fractional Λ-space, consider the function

$$f(x)=x^3 \quad , \tag{2.1}$$

then according to Eq.(1.9b),

$$F(x) = \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{s^3}{(x-s)^\gamma} ds = \frac{6\, x^{4-\gamma}}{\Gamma(5-\gamma)} \tag{2.2}$$

Nevertheless, we may formulate the fractional Λ-space if we substitute x from Eq.(1.6a). Indeed,

$$F(X) = \frac{6(\Gamma(3-\gamma)X)^{\frac{4-\gamma}{2-\gamma}}}{\Gamma(5-\gamma)} \quad . \tag{2.3}$$

Figs. (1,2) clarify the original and the fractional Λ-space. Fig.1. corresponds to

Eq.(2.01), while Fig.2 corresponds to Eq.(2.3) for γ=0.8.

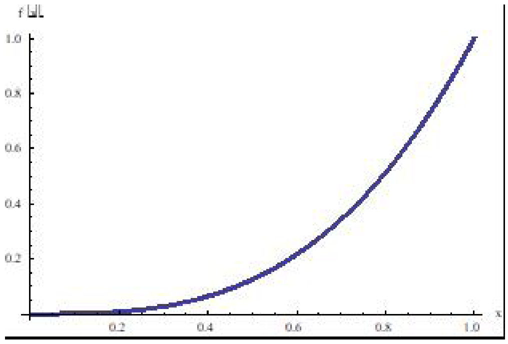

Figure 1: The curve f(x)=x^3 in the initial space

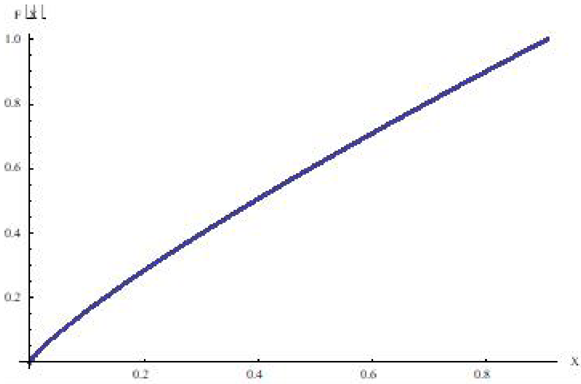

Figure 2: The curve in the Λ-space for γ=0.8

Further, the Λ-fractional derivative is defined only in the Λ-fractional space as the conventional derivative,

$$ {}_{a}^{\Lambda}D_{x}^{\gamma} f(x) = \frac{dF(X)}{dX} \quad . \tag{2.4}$$

Therefore, for the present case of f(x)=x^3, the corresponding ${}_{a}^{\Lambda}D_{x}^{\gamma} f(x)$ in the Λ-fractional space is defined by,

$$ {}_{a}^{\Lambda}D_{x}^{\gamma} f(x) = 0.903\, X^{-0.167}. \tag{2.5}$$

Fig.3 shows the curve of F(X) ${}_{a}^{\Lambda}D_{x}^{\gamma} f(x)$ in the Λ-fractional space.

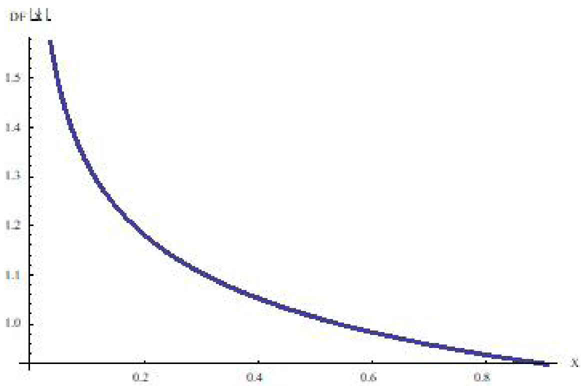

Figure 3: The derivative ${}_{a}^{\Lambda}D_{x}^{\gamma} f(x)$ in the Λ-fractional space

Transferring the function of the ${}_{a}^{\Lambda}D_{x}^{\gamma} f\left(x\right)$ in the initial space with,

$$X=0.908x^{1.2} \tag{2.6}$$

and

$$f^{1}(x)=\frac{1}{\Gamma(0.8)}\frac{d}{dx}\int_{0}^{x}\frac{0.903(0.908s^{1.2})^{-0.167}}{(x-s)^{0.2}}ds\ . \tag{2,7}$$

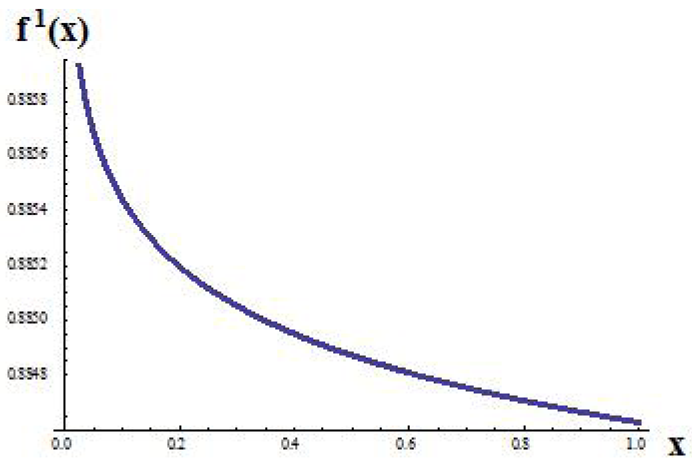

Figure: 4 The function $f^{1}(x)$ of the Λ-fractional derivative in the initial space for γ=0.8

It has been pointed out that the present Λ-fractional analysis is straight forward when 0<γ<1. That is quite evident, since the Riemann-Lowville fractional derivatives for k-1<γ< k are defined as:

$${}_{a}^{RL}D_{x}^{\gamma}f\left(x\right)=\frac{d^{k}}{dx^{k}}{}_{a}I_{x}^{k-\gamma}f\left(x\right) \tag{2.8}$$

Therefore, an analogue Λ-fractional derivative for 1<γ is not possible, since the fraction

$$\frac{\frac{d^{k}}{dx^{k}}{}_{a}I_{x}^{k-\gamma}f(x)}{\frac{d^{k}}{dx^{k}}{}_{a}I_{x}^{k-\gamma}(x)} \tag{2.9}$$

cannot be represented as a derivative of the type, $\frac{d^{k}\Phi(X)}{dX^{k}}$

The present fractional analysis for 0<γ<1 is extended just to include fractional derivatives with 1< γ.

The Λ-Fractional Analysis for k-1<γ<k

The Λ-fractional analysis is extended in the present paragraph, just to include fractional derivatives higher than 1. Let us consider the order γ of the Λ-fractional derivative where k-1<γ< k. Introducing a Λ-fractional derivative of order

$$\gamma_o = \frac{\gamma}{k}, \qquad 0<\gamma_o<1 \qquad , \tag{3.1}$$

the Λ-fractional space is defined by (X, F(X)) with,

$$X = {}_a I_x^{1-\gamma_o} x, \qquad F(X) = {}_a I_x^{1-\gamma_o} f(x(X)). \tag{3.2}$$

Hence, the Λ-fractional derivative in the Λ-fractional space is defined as the k-th derivative of the function F(X) concerning X, i.e.

$${}_a^\Lambda D_x^\gamma f(x) = \frac{d^k F(x(X))}{dX^k} = G(X) \qquad . \tag{3.3}$$

Transferring the function G(X) from the Λ-fractional space to the initial space the Λ-fractional derivative g(x) has been derived for k-1<γ<k with,

$$g(x) = {}_a^{RL}D_x^{1-\gamma_o} G(X(x)). \tag{3.4}$$

Therefore, the Λ-fractional derivative has been extended just to include orders higher than one.

$$f^\gamma(x) = {}_a^{RL}D_x^{1-\gamma_o} G(X(x)) = {}_a^{RL}D_x^{1-\gamma_o} I_x^{1-\gamma_o} f(x). \tag{3.5}$$

For a better explanation between the initial and the fractional Λ-space, consider the function $f(x)=x^3$ and the order γ=2.4. According to Eq.(3.1), $\gamma_o=0.8$. Hence the function F(X) in the Λ-fractional space is defined by,

$$F(x) = \frac{1}{\Gamma(1-\gamma o)} \int_a^x \frac{s^3}{(x-s)^{\gamma o}} ds = \frac{6\, x^{4-\gamma o}}{\Gamma(5-\gamma o)} \tag{3.6}$$

Nevertheless, we may formulate the fractional Λ-space if we substitute x from Eq.(3.2). Indeed,

$$F(X) = \frac{6(\Gamma(3-\gamma o)X)^\delta}{\Gamma(5-\gamma o)} \qquad . \tag{3.7}$$

where, $\delta = \frac{4-\gamma o}{2-\gamma_o} = \frac{3.2}{1.2} = 2.67.$

Therefore, the problem of defining the Λ-fractional space for the present case has been transferred to the preceding paragraph where the function $f(x)=x^3$ with the fractional order γ=0.8. Hence,

$${}_a^\Lambda D_x^{2,4} f(x) = \frac{d^3 F(x(X))}{dX^3} = 2,993\, X^{-0.333} = G(X) \cdot \tag{3.8}$$

The third derivative of the function F(X), in the Λ-space, is given by G(X),

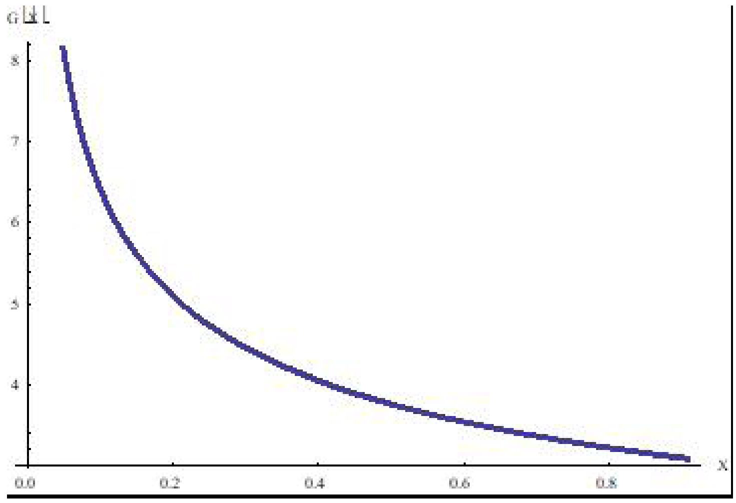

Figure: 5 The ${}_a^{\Lambda}D_x^{2,4} f\left(x\right)$ in the Λ-fractional space of t6he function f(x0=x³

Transferring G(X) into the initial space, the Λ-fractional derivative ${}_a^{\Lambda}D_x^{2,4} f\left(x\right)$ is defined by,

$$g(x)=\frac{1}{\Gamma(\gamma_0)}\frac{d}{dx}\int_o^x \frac{G(X(s))}{(x-s)^{1-0.8}}\mathrm{ds}. \tag{3.9}$$

In fact

$$g(x)=\frac{1}{\Gamma(0.80)}\frac{d}{dx}\int_0^x \frac{2.993\left(0.907s^{1.2}\right)^{-0.33}}{(x-s)^{0.2}}ds. \tag{3.10}$$

and

$$f^{2.4}\ (\mathrm{x})=2.084x^{-0.6}\ . \tag{3.11}$$

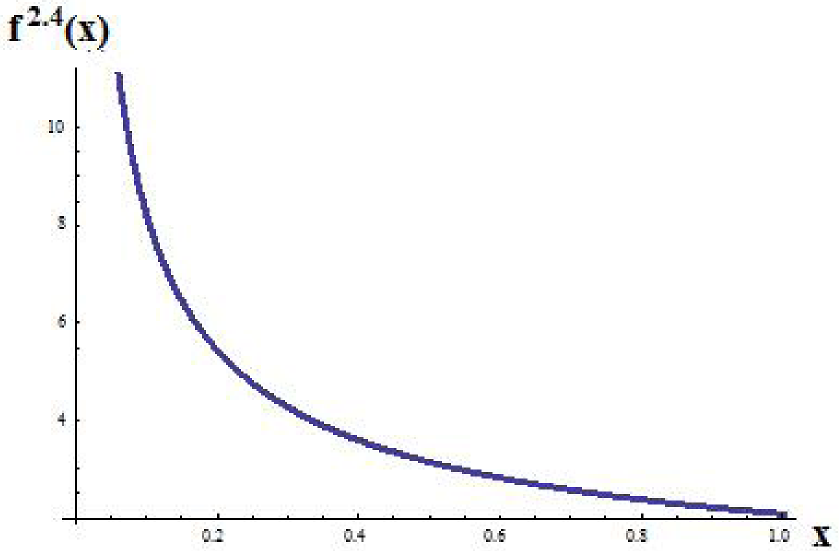

Figure: 6 The Λ-fractional derivative ${}_a^{\Lambda}D_x^{2,4} f\left(x\right)$ in the initial space of the function *f(x)=x³*

The procedure presented in the present paragraph will be extended to the Dirac delta function in the next paragraph.

The Λ-fractional Dirac Delta Function

The Dirac delta function is defined by,

$$\int_R f(y)\,\delta(x-y)\,dy = f(x)\,. \qquad (4.1)$$

Further, for integer n,

$$\int_R \frac{d^n\delta(y-x)}{dy^n} f(y)\,dy = (-1)^n \frac{d^n f(x)}{dx^n} \qquad (4.2)$$

The question is about theΛ-fractional Dirac delta function when n is rational. In that case, two types are discussed. First when 0<n<1 and secondly when 1<n. In the first type, the Λ-fractional space for γ=n is formulated. Then, according to Eq.(4.1),

$$\int_R F(Y)\,\delta(X-Y)\,dY = F(X) \qquad (4.3)$$

Then the steps for transferring F(X) into the initial space f(x) are followed exactly as they were described in the previous chapters. Nevertheless, when k-1<γ<k, the procedure for defining the Λ-fractional function is quite different, since the method for the corresponding Λ-fractional derivatives for 1<γ is followed. Indeed, the γ_o=γ/k order is considered. Hence the Λ-fractional analysis for the γ_o order is considered. Then the Λ-fractional derivative of γ order is transformed into the k-th derivative in the Λ-fractional space. The function of the k-th derivative $\frac{d^k F(X)}{dX^k}$ is transferred into the initial space, taking into consideration the order γ_o. The procedure will be explained by applying it to the the Λ-fractional derivative of the unit of γ-order.

The procedure will be explained by discussing the $\frac{d^{0.8}\delta(x-y)}{dx^{0.8}}[1]$ and $\frac{d^{2.4}\delta(x-y)}{dx^{2.4}}[1]$ cases.

Λ-fractional Dirac Delta Function of Order γ=0.8

Consider the function f(x)=1 in the initial space. Transferring the function f(x) into the Λ-fractional space (X,F(X)) with fractional order γ=0.8,

$$F(x)=\frac{1}{\Gamma(1-0.8)}\int_0^x \frac{1}{(x-s)^{0.8}}\,ds = 1.089x^{0.2}. \qquad (4.4)$$

Further, the X variable in the Λ-fractional space is defined by,

$$X=\frac{1}{\Gamma(1-0.8)}\int_0^x \frac{s}{(x-s)^{0.8}}\,ds=0.908x^{1.2}. \qquad (4.5)$$

Therefore, $x=\left(\frac{X}{0.91}\right)^{\frac{1}{1.2}}$.

Substituting x into the function F(x), the function F(X) is defined into the Λ-fractional space by,

$$F(X)=1.106X^{0.167}. \tag{4.6}$$

The function *F(X)* in the Λ-fractional space corresponds to *f(x)*=1 in the initial space.

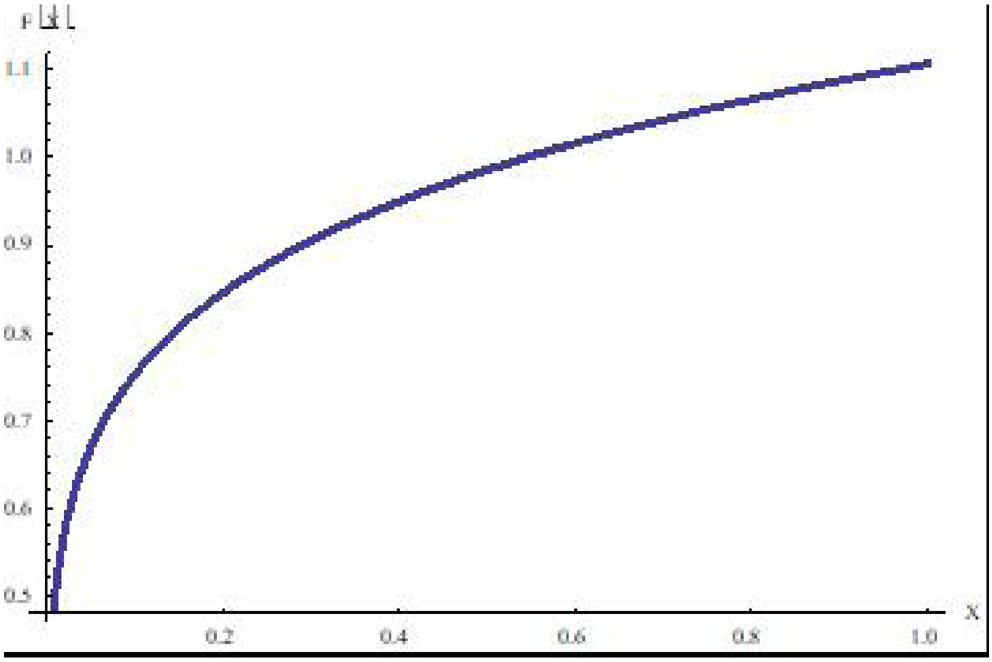

Figure: 7 The function *F(X)* in the Λ-space corresponding to *f(x)*=1 in the initial space.

Then, according to Eq.(4.1),

$$\int_R \frac{d\delta(Y-X)}{dY} F(Y)\, dY = (-1)\frac{dF(X)}{dX} = -D[F(X)]. \quad (4.7)$$

Thus, for the function F(X) in the Λ-fractional space corresponding to the initial function f(x)=1, the derivative of the Dirac delta function yields in the Λ-fractional space,

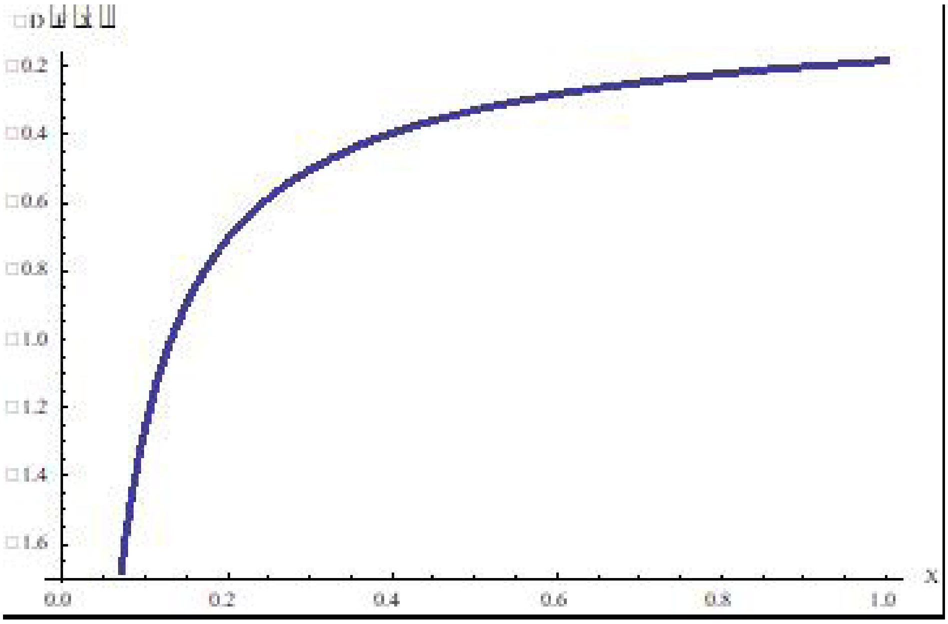

Figure: 8 The first derivative of the Dirac Delta function corresponding to F(X) in the Λ-fractional space

The next step is to transfer the function –D[F(X)], as it is defined in theΛ-fractional space and indicated in Fig.8, into the initial space.

$$d^{0.8}f(x)=\frac{1}{\Gamma(0.8)}\frac{d}{dx}\int_o^x -\frac{D(X(s))}{(x-s)^{1-0.8}}. \tag{4.8}$$

Using numerical methods, the function $d^{0.8}f(x)$ corresponding to $\frac{d^{0.8}(x-y)}{dx^{0.8}}[1]$.

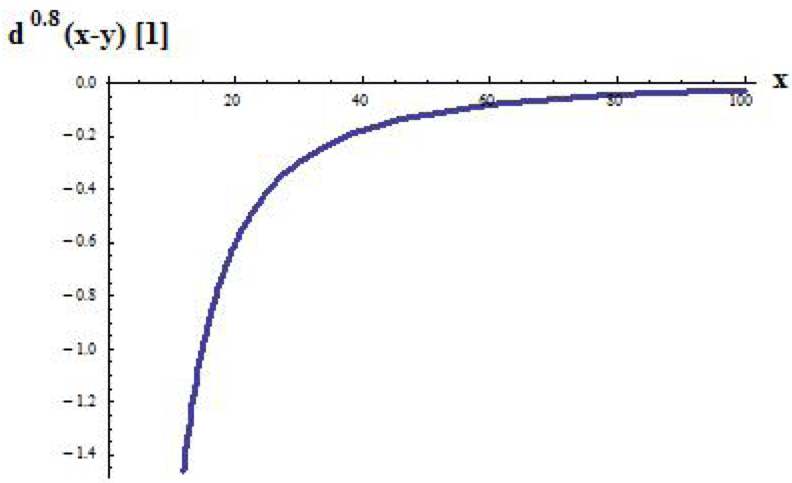

Figure: 9 The function of the Dirac Delta of 1 of order γ=0.8

Λ-fractional Dirac Delta Function of Order γ=2.4

The present section is devoted to the definition of the Dirac Delta function of fractional order *1<γ=2.4*. As was pointed out in the preceding sections, the present γ order lies in the interval *2<γ=2.4<3*. Hence, the order γ_o=2.4/3=0.8 lying in the interval (0,1) is considered. Consequently, the Λ-fractional analysis is formulated with fractional order γ_o=0.8<1. Thus, the function *f(x)*=1 is transferred into the Λ-fractional space as the function *F(X)* defined by Eq.(4.6). Then, the fractional derivative of the function *f(x)*=1 of order *γ*=2.4 is transferred, into the Λ-fractional space of the order γ_o=0.8, as the derivative,

$$\frac{d^3(Y-X)}{dY^3}F(Y) = (-1)\frac{d^3F(X)}{dX^3} \quad . \tag{4.9}$$

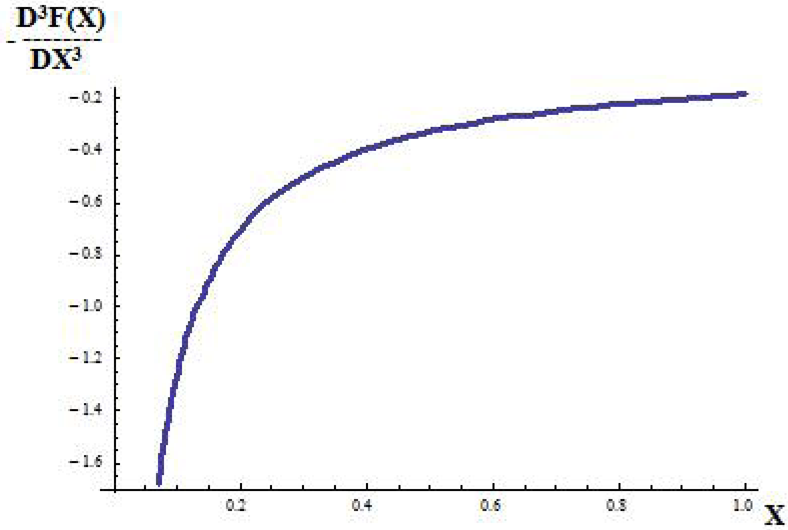

Figure: 10 The curve of ${}_0^{\Lambda}D_x^{2.4}[1]$ in the Λ-fractional space

Transferring that curve from the Λ-fractional space to the initial one, the ${}_0^{\Lambda}D_x^{2.4}[1]$ in the initial space should be,

$$d^{2.4}f(x)=\frac{1}{\Gamma(0.8)}\frac{d}{dx}\int_o^x -\frac{\frac{D^3(X(s))}{DX^3}}{(x-s)^{1-0.8}}.ds \tag{4.10}$$

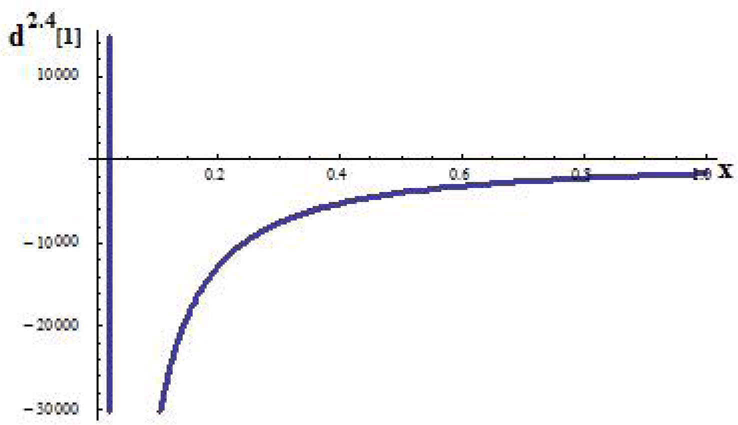

Figure: 11 shows the 2.4 derivative of 1 in the initial space.

Conclusion–Further Results

The Λ-fractional analysis has been completed just to include fractional orders $\gamma>1$. Hence a complete Λ-Fractional analysis has been presented, ready to offer results in any applications. In addition, the Λ-fractional analysis of the Dirac Delta function offers strong tools in various applications, like visco-elastic problems [8-11], earthquake studies [13], and not only. Further applications may be found in [19-21].

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