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About the strong EULER-GOLDBACH conjecture

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Abstract

In this paper, a "local" algorithm is determined for the construction of two recurrent sequences of positive primes (U_{2n}) and (V_{2n}), ((U_{2n}) dependent on (V_{2n})), such that for each integer n ≥ 2 , their sum is equal to 2n. To form this, a third sequence of primes (W_{2n}) is defined for any integer n ≥ 3 by : $W_{2n} = Sup(\ p \in \mathcal{P}\ : p \leq 2n - 3)$, where $\mathcal{P}$ is the infinite set of primes. Goldbach's conjecture has been proved for all even integers 2n between 4 and 4.10^{18}. In the table of terms of Goldbach sequences given in appendix 10 , values of the order of $2n = 10^{1000}$ are reached.This "ascent and descent " algorithm proves Goldbach's conjecture ; an analogous proof by recurrence is established and an increase of U_{2n} by $0.7(ln(2n))^{2.2}$ is established. Moreover, the Lagrange-Lemoine-Levy conjecture and its generalization, the Bezout-Goldbach conjecture, are proven by the same type of procedure.

Keywords: Prime numbers, prime number theorem, weak and strong Goldbach conjectures, Bertrand-Chebyshev theorem, gaps between consecutive primes, Lagrange-Lemoine-Levy conjecture, Bezout-Goldbach conjecture.

1 Background

Number theory, “ the queen of mathematics“ deals with structures and properties defined on integers and primes (see Euclid [11], Hadamard [13], Hardy & Wright [14], Landau [20]). Numerous problems have been raised and conjectures made, the statements of which are often simple but very dificult to prove. These main components include:

Elementary arithmetic

* Determination and properties of primes.
* Operations on integers (basic operations, congruence, gcd, lcm, ………..).
* Decomposition of integers into products or sums of primes (fundamental theorem of arithmetic, decomposition of large numbers, cryptography, and Goldbach's conjecture) .

Analytical number theory :

* The Riemann hypothesis.
* Distribution of primes (Prime number theorem, Hadamard [13], De la Vallée-Poussin [33], Littlewood [23] and Erdos [10]).
* Gaps between consecutive primes, (Bombieri & Davenport, [3], Cramer [8], Baker, Harmann, Iwaniec & Pintz [4], [5],[18] , Granville [12], Shanks [27], Tchebychev [32] and Zhang [36]).

Algebraic, probabilistic, combinatorial and algorithmic number theories.

* Modular arithmetic, diophantine approximations, equations.
* Arithmetic functions and algebraic geometry.

2 Definitions, notations and reminders

(2.1) The integers n, k, p, q, r,…….. are always positive.

(2.2) Let $\mathcal{P}$ the infinite set of positive primes (called simply primes) :

$$\mathcal{P} = \{ p_k (k \in \text{IN*}) : p_k \text{ is the kth positive prime} ; (p_k < p_{k+1} \text{ and } \lim p_k = +\infty)\}$$

$$(p_1 = 2 ; p_2 = 3 ; p_3 = 5 ; p_4 = 7 ; p_5 = 11 ; p_6 = 13 ; ....).$$

(2.3) The writing of large numbers (see appendix 10) is simplified using the following constants :

a) $M = 10^9$

b) $R = 4.10^{18}$

c) $G = 10^{100}$

d) $S = 10^{500}$

e) $T = 10^{1000}$

(2.4) ln(x) denotes the neperian logarithm of the strictly positive real x , (x > 0).

(2.5) Let (W_{2n}) be the sequence of primes defined by :

(2.5.1) For any integer n ≥ 3 , $W_{2n} = \text{Sup}(p \in \mathcal{P} : p \leq 2n-3)$

(2.6) Any sequence denoted by $(G_{2n}) = (U_{2n} ; V_{2n})$ verifying the property :

“For any integer n ≥ 2 , U_{2n} and V_{2n} are primes and $U_{2n} + V_{2n} = 2n$” , is called a **<u>Goldbach sequence.</u>**

(2.7) Iwaniec & Pintz [18] have shown that for any integer $n \in \mathbb{N} + 3$, there is always a prime between $n - n^{23/42}$ and n .
Baker & Harman [4], [5] concluded that for any sufficiently large integer n there is a prime in the interval $[n ; n + o(n^{0.525})]$. Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form :

(2.7.1) $\forall \varepsilon > 0$, $\exists k_\varepsilon \in \mathbb{N}^*$ / $\forall$ k $\in \mathbb{N}^*$, (k > k_ε), $p_{k+1} - p_k < \varepsilon.p_k^{0.525}$

(2.8) According to the Cramer-Maier-Nicely conjecture [1], [3], [8], [12], [24], [25], for any real c > 2 , for any integer k ≥ 500 ,

(2.8.1) $p_{k+1} - p_k \le 0.7(ln(p_k))^c$ **(with probability one).**

3 Introduction

Chen [6], Hardy & Littlewood [15], Hegfollt [16], Ramaré & Saouter [26], Tao [31], Tchebychev [32] and Vinogradov [34] have taken important steps and obtained promising results on Goldbach's conjecture.

Indeed, Helfgott & Platt proved Goldbach's weak conjecture in 2013. Silva, Herzog & Pardi [29] held the record for calculating the terms of Goldbach sequences after determining pairs of primes $(U_{2n} ; V_{2n})$ verifying :

(3.1) For any integer n , $(4 \le 2n \le 4.10^{18})$: $(U_{2n} + V_{2n} = 2n)$.

In previous research work, there is no explicit construction of recurrent sequences of Goldbach primes of the form :

$(G_{2n}) = (U_{2n} ; V_{2n})$ satisfying for any integer n ≥ 2 the equality : $(U_{2n} + V_{2n} = 2n)$.

In this article, two sequences of primes are developed using a simple and efficient algorithm to compute for any integer n ≥ 3 by successive iterations any term U_{2n}^{2n} and V_{2n}^{2n} of a Goldbach sequence. Using Maxima scientific software on a personal computer, Silva's record is broken, and the values $2n = 10^{500}$ and even $2n = 10^{1000}$ are reached. The proof of Goldbach's conjecture can be established on the same principle, using reasoning by recurrence. Moreover, the Lagrange-Lemoine-Lévy conjectures [9], [17], [19], [24], [25], [30], [35] and its generalization, the Bezout-Goldbach conjecture are validated. Using case disjunction reasoning, we construct two recurrent sequences of primes (V_{2n}) and (U_{2n}) according to the sequence (W_{2n}) by the following process. For any integer n ≥ 2 ,

(3.2) $(U_4 = 2 ; V_4 = 2)$

Let n be an integer , (n ≥ 3) :

1. Either, $(2n - W_{2n})$ is a prime, then U_{2n} and V_{2n} are defined directly in terms of W_{2n} .

2. Either, $(2n - W_{2n})$ is a composite number, then V_{2n} and U_{2n} are defined from the preceding terms of the sequence (G_{2n}).

4 Methodology

To determine pairs of primes that verify Goldbach's conjecture, three sequences of primes (W_{2n}), (V_{2n}), (U_{2n}) are defined and verify the following properties :

(4.1) $\lim V_{2n} = +\infty$.

(4.2) For any integr n ≥ 2 , V_{2n} is defined as a function of $W_{2n} = \text{Sup}(p \in \mathcal{P} : p \le 2n - 3)$.

(4.3) (W_{2n}) is an increasing sequence that contains all primes except $p_1 = 2$.

(4.4) $\lim W_{2n} = +\infty$.

(4.5) (U_{2n}) is a complementary sequence of negligible primes with respect to 2n , $(U_{2n} \ll 2n)$.

(4.6) For any integer n ≥ 3,
If $(2n - W_{2n})$ is a prime " special case ", then V_{2n} and U_{2n} are defined by :

(4.7) $V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$

Otherwise, if $(2n - W_{2n})$ is a composite number " general case ",
we use the previous terms of the sequence (G_{2n}). So we look for an integer k to obtain two terms $U_{2(n-k)}$ and $V_{2(n-k)}$ satisfying the following conditions :

(4.8) $U_{2(n-k)}$, $V_{2(n-k)}$ and $U_{2(n-k)}+2k$ are primes $U_{2(n-k)}+V_{2(n-k)}=2(n-k)$

(which is always possible ; see the proof in Theorem 5).

Thus, by setting :

(4.9) $V_{2n}=V_{2(n-k)}$ and $U_{2n}=U_{2(n-k)}+2k$

two new primes V_{2n} and U_{2n} satisfying (4.10) are generated.

(4.10) $U_{2n}+V_{2n}=2n$.

This process is then repeated, incrementing n by one unit : (n → n+1).

5 Theorem

There exists a recurrent Goldbach sequence of primes $(G_{2n})=(U_{2n}\ ;\ V_{2n})$ satisfying for any integer $n \geq 2$:

U_{2n} and V_{2n} are primes and their sum is equal to 2n.

(5.1) $(U_{2n}, V_{2n} \in \mathcal{P}$ and $U_{2n}+V_{2n}=2n)$.

(5.2) An algorithm can be used to explicitly compute any term U_{2n} and V_{2n} .

Proof of Theorem 5

<u>First Method :</u>

For any integer $n \geq 3$,

If $(2n - W_{2n})$ is a prime, then V_{2n} and U_{2n} are defined by :

(5.3) $V_{2n}=W_{2n}$ and $U_{2n}=2n-W_{2n}$

Otherwise,

if $(2n - W_{2n})$ is a composite number ,

we use the previous terms of the sequence (G_{2n}) to determine (U_{2n}) and (V_{2n}) .

For any integer q such that : $(1 \leq q \leq n-3)$, we have : $3 \leq U_{2(n-q)} \leq n$.

For any integer k such that $(4 \leq 2k \leq n-1)$, there are two primes p_m and p_r , (m > r) in the interval [*4 ; n*] such that :

(5.4) $p_m - p_r = 2k$

(see Bombieri & Davenport [1], Cramer [8], Iwaniec & Pintz [18] , Tchebychev [32]).

Then there is an integer k verifying , $(4 \leq 2k \leq n-3)$ such that :

(5.5) $R_{2n}=U_{2(n-k)}+2k$ is a prime

The smallest integer k denoted k_n such that R_{2n} is a prime is chosen. So let :

(5.6) $U_{2n}=U_{2(n-k_n)}+2k_n$ and $V_{2n}=V_{2(n-k_n)}$

(These two terms are primes)

In the previous steps two primes, $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to $2(n-k_n)$ were determine

(5.7) $U_{2(n-k_n)}+V_{2(n-k_n)}=2(n-k_n)$

By adding the term k_n to each member of the equality (5.6), it follows :

(5.8) $U_{2(n-k_n)}+2k_n+V_{2(n-k_n)}=2(n-k_n)+2k_n$

(5.9) ⇔ $\{U_{2(n-k_n)}+2k_n\}+V_{2(n-k_n)}=2n$

(5.10) ⇔ $U_{2n}+V_{2n}=2n$

Finally, for any integer n ≥ 3 , this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.

Second Method :

The demonstration can be made using the following strong recurrence principle.
Let P(n) be the following property defined for any integer n ≥ 2 by :
P(n) : " For any integer p satisfying : (2 ≤ p ≤ n) , there exists two primes U_{2p} and V_{2p} and such their sum is equal to 2p : ($U_{2p} + V_{2p}$ = 2p) " .

Let us show by strong recurrence that P(n) is true for any integer n ≥ 2.
a) P(2) is true : it suffices to choose $U_4 = V_4 = 2$.
b) Let us show that the property P(n) is hereditary : (i.e for any integer n ≥ 2 P(n) ⇒ P(n+ 1))

Assume property P(n) is true,
If (2(n +1 - $W_{2(n+1)}$) is a prime, then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by :

(5.11) $V_{2(n+1)} = W_{2(n+1)}$ and $U_{2(n+1)}$ = 2(n+1) - $W_{2(n+1)}$

Otherwise, if (2(n+1) - $W_{2(n+1)}$) is a composite number ,
there exists an integer k to obtain two terms $U_{2(n+1-k)}$ and $V_{2(n+1-k)}$ satisfying the following conditions :

(5.12) $U_{2(n+1-k)}$, $V_{2(n+1-k)}$ and $U_{2(n+1-k)}$ + 2k are primes.

$U_{2(n+1-k)} + V_{2(n+1-k)}$ = 2(n+1-k) (which is always possible ; see first method).

Thus, by setting :

(5.13) $V_{2(n+1)} = V_{2(n+1-k)}$ and $U_{2(n+1)} = U_{2(n+1-k)}$ + 2k.

two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying ($U_{2(n+1)} + V_{2(n+1)}$ = 2(n+1)) are generated. It follows that P(n+1) is true, then the property P(n) is hereditary : (P(n) => P(n+1)).

Therefore, for any integer n ≥ 2 the property P(n) is true ; it follows that :

∀ n ≥ 2 there are two primes U_{2n} and V_{2n} and such their sum is 2n : ($U_{2n} + V_{2n}$ = 2n)

6 Lemma

The sequence (U_{2n}) verifies the following estimation : For any integer n ≥ 65 ,

(6.1) $U_{2n} \leq (2n)^{0.55}$

Proof of Lemma 6

According to the programm 9.2 and appendix 10, the estimate (6.1) is verified for any integer n such that : (65 ≤ n ≤ 2000) . For any integer n > 2000 , the proof is established by recurrence. For this purpose, let P_1(n) be the following property :

(6.2) P_1(n) : " There exists a strictly increasing sequence of positive numbers (C_n) such that : $U_{2n} \leq C_n (2n)^{0.525}$ " .

a) P_1(2000) is true according to program 9.2 and the table in appendix 10.

b) For any integer n ≥ 2000, let us show that P_1(n) is hereditary , (i.e $P_1(n) \Rightarrow P_1(n+1)$)

Assume that P_1(n) is true : then,
If (2(n+1) - $W_{2(n+1)}$) is a prime, then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by :

(6.3) $$V_{2(n+1)} = W_{2(n+1)} \quad \text{and} \quad U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$$

According to the results in [4], [5], [18] there is a constant K , (0 < K < 1) /

$$2(n+1) - K(2(n+1))^{0.525} < W_{2(n+1)} < 2(n+1)$$

$$\Rightarrow \quad U_{2(n+1)} < K(2(n+1))^{0.525}$$

$$\Rightarrow \quad U_{2(n+1)} \le C_{n+1}(2(n+1))^{0.525}$$

Otherwise, if (2(n+1) - $W_{2(n+1)}$) is a composite number ,

(6.4) $$\exists\, p \in \mathbb{N}^* \;/\; U_{2(n+1)} = U_{2(n+1-p)} + 2p$$

U < ( )

According to [4], [5], [18], the smallest integer p defined in (6.4) verifies:

(6.5) $$2p < KU_{2(n+1-p)}^{0.525} \quad \text{and} \quad U_{2(n+1-p)} < C_{n+1-p}(2(n+1-p))^{0.525}$$

It follows :

$$U_{2(n+1)} < KC_{n+1-p}^{0.525}(2(n+1-p))^{0.275625} + C_{n+1-p}(2(n+1-p))^{0.525}$$

Then,

(6.6) $$U_{2(n+1)} < C_{n+1}.(2(n+1))^{0.525}$$

and, by setting : $C_n = (2n)^{0.025}$, it follows :

(6.7) $$U_{2(n+1)} < (2(n+1))^{0.55}$$

P_1(n+1) is true then P_1(n) is hereditary. So for any integer n ≥ 2000, the property P_1(n) is true.

(The inequality (6.7) is verified with the aid of the scientific software Maple studying the functions of the type :

f : x → a$x^{0.275625}$ + b$x^{0.525}$ increased by g : x → $x^{0.55}$, (a , b > 0)).

* **Remark :** A more precise mark-up can be obtained using the Cippola or Axler frames. [7], [2].

7 Theorem

For any integer n ≥ 3 , it is easy to check :
7.1 (W_{2n}) is a positive increasing sequence of primes.
7.2 { W_{2n} : n ∈ IN* } ∪ {2} = $\mathcal{P}$
7.3 lim W_{2n} = +oo
7.4 (V_{2n}) is a sequence of primes.
The following results are validated with **probability one :**
7.5 n ≤ $V_{2n} \le W_{2n}$
7.6 3 ≤ $2n - W_{2n} \le U_{2n}$ ≤ n
7.7 lim V_{2n} = +oo

Proof of Theorem 7

7.1 For any integer $n \geq 2$ let A_n be the following set : $A_n = \{ p_k \in \mathcal{P} : p_k \leq 2n - 3 \}$.
$A_n \subset A_{n+1}$ therefore, $W_{2n} \leq W_{2(n+1)}$, so the sequence (W_{2n}) is a positive increasing sequence of primes.

7.2 Any prime except $p_1 = 2$ is odd, hence the result.

7.3 $\lim W_{2n} = \lim p_n = +oo$.

7.4 By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n - 2$ such that : $V_{2n} = V_{2(n-k)}$; so, by reccurence the terms of the sequence (V_{2n}) are primes ; moreover, there exists a strictly increasing sub-sequence (V'_{2n}) of (V_{2n}) verifying $\lim (V'_{2n}) = +oo$.

7.5 According to Lemma 6, for any integer $n \geq 65$, $U_{2n} < (2n)^{0.55}$; therefore,

$$U_{2n} < (2n)^{0.55} < n \quad \text{and} \quad V_{2n} = 2n - U_{2n} > 2n - n > n.$$

For any integer n / ($3 \leq n \leq 65$), verification is carried out according to the program in 9.2 and the table in appendix 10.

7.6 According to 7.5,

$$n \leq V_{2n} \Rightarrow U_{2n} = 2n - V_{2n} \leq 2n - n \leq n \text{ ; moreover,}$$
$$V_{2n} \leq W_{2n} \Rightarrow 2n - W_{2n} \leq 2n - V_{2n} = U_{2n} .$$

7.7 By 7.5, for any integer $n \geq 2$, $n \leq V_{2n}$, so $\lim (V_{2n}) = +oo$.

8 Remarks

8.1 There are infinitely many integers n such that : $U_{2n} = 3, 5, 7$ or 11.

8.2 $V_{2n} \sim 2n$ for ($n \rightarrow +oo$) .

8.3 For any sufficiently large integer n , ($n \geq 5000$) : $U_{2n} \ll V_{2n}$ and $\lim(U_{2n} / V_{2n}) = 0$.

8.4 The smallest integer n such that : $U_{2n} \neq 2n - W_{2n}$ is obtained for n = 49 and G_{98}= (79 ; 19).

(This type of term increases in the Goldbach sequence (G_{2n}) as n increases, in the sense of the Schnirelmann density, and there are an infinite number of them; their proportion per interval can be computed using the results given in [28]).

8.5 If $q \geq 5$ is an odd integer, we could generalize this algorithm with sequences (W'_{2n}) defined by :

(8.6.1) $\forall n \in \mathbb{N}, (n \geq q + 1.5)$ $\qquad W'_{2n} = \text{Sup}(p \in \mathcal{P} : p \leq 2n - q)$.

Other Goldbach sequences (G'_{2n}) independent of (G_{2n}) are thus generated.

8.6 The sequence (G_{2n}) is extremal in the sense that for any integer $n \geq 2$, V_{2n} and U_{2n} are the largest and smallest possible primes such that : $U_{2n} + V_{2n} = 2n$.

8.7 The Cramer-Maier-Nicely conjecture [8], [12], [17], [19], [21], [22], [24], [25], [30] is verified with probability one. It leads to the following mark-up : For any integer $p \geq 500$,

(8.7.1) $$U_{2p} \leq 0.7(\ln(2p))^{(2.2 - 1/p)}$$ **(with probability one)**

The proof is similar to that of Lemma 6, using the same type of reasoning by recurrence, validated by the study of functions of the type : $f : x \rightarrow ag(x) + b(\ln(g(x)))^c$ (a,b > 0) , (c > 2) , (with $g : x \rightarrow 0.7(ln(x))^{(c - 1/x)}$) and $h : x \rightarrow 0.7(ln(x))^{(2.2 - 1/x)}$ using Maple software.

* **Remark :** A better mark-up can be obtained via [24], [25], [27] .

8.8 According to Bombieri [3] and using the same method as in the proof of Lemma 6 , we obtain the following estimate of U_{2n} :

(8.8.1) $\forall \varepsilon > 0$, $\qquad U_{2n} = O((ln(2n))^{1.3+\varepsilon})$, **(on average).**

9 Algorithm

Algorithm written in natural language

Inputs

Input four integer variables : k, N, n, P.

Input : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$,……………………………… , p_N the first N primes. :
Input : n=3.
Input : P = M, R, G, S or T as indicated in paragraph 2.

Algorithm body

A) Compute : $W_{2n} = \text{Sup}(\ p \in \mathcal{P}\ :\ p \leq 2n-3\ )$

If $T_{2n} = (2n - W_{2n})$ is a prime,
Let :
(9.1.1) $U_{2n} = T_{2n}$ and $V_{2n} = W_{2n}$
otherwise ,

B) If T_{2n} is a composite number, let : k=1.

B.1) While $U_{2(n-k)} + 2k$ is a composite number,

assign to k the value : k+1 ($k \rightarrow k + 1$).

Return to B1)
End While .

Assign to k the value : ($k \rightarrow k_n$)

(9.1.2) Let : $U_{2n} = U_{2(n-k_n)} + 2k_n$ and $V_{2n} = V_{2(n-k_n)}$

Assign to n the value n+1 , ($n \rightarrow n + 1$) and return to A)
End :
Outputs for integers less than 10^4 : Print (2n = ... ; 2n - 3 = ... ; $W_{2n} = \ldots$; $T_{2n} = \ldots$; $V_{2n} = \ldots$; $U_{2n} = \ldots$).
Outputs for large integers : Print (2n - P = ... ; 2n - 3 - P= ... ; W_{2n} - P = ... ; $T_{2n} = \ldots$; $V_{2n} - P = \cdots$; $U_{2n} = \ldots$).

9.2 Program written with Maxima software for $2n = 10^{500}$

```
r:0 ; n1:10**500 ; for n:5*10**499+10000 thru 5*10**499+10010 do

( k:1 , a:2*n , c:a-3 , test:0 , b:prev_prime(a-1) ,

if primep(a-b)

then print(a-n1,c-n1,b-n1,a-b,b-n1,a-b)

otherwise ( r:r+1 ,

while test=0 do

( if ( primep(c) and primep(a-c) )

then ( test:1 , print(a-n1,a-n1-3,b-n1,a-b,c-n1,a-c," Ret ",r))

else ( test:0 , c:c-2*k ))) ) ;
```

10 Appendix

Application of Algorithm 9 :
Table of U_{2n} and V_{2n} terms of the Goldbach sequence (G_{2n}) computed from program 9.2 , ($2 \leq 2n \leq 10^{1000} + 4020$).

The " ** sign" in the table below indicates the results given by the algorithm 9 in case B) of return to the previous terms of the sequence (G_{2n}). **<u>WATCH OUT !</u>** For large integers n ($2n > 10^9$) , to simplify the display of large numbers, the results are entered as follows : 2n - P , (2n - 3) - P , W_{2n} - P , T_{2n}, V_{2n}- P and U_{2n} with , P = M, R, G, S, or T constants defined in (2.3).

2n	**2n - 3**	W_{2n}	**T_{2n} = 2n - W_{2n}**	**V_{2n}**	**U_{2n}**
4	1	x	x	2	2
6	3	3	3	3	3
8	5	5	3	5	3
10	7	7	3	7	3
12	9	7	3	7	5
14	11	11	3	11	3
16	13	13	3	13	3
18	15	13	5	13	5
20	17	17	3	17	3
22	19	19	3	19	3
24	21	19	5	19	5
26	23	23	3	23	3
28	25	23	5	23	5
30	27	23	7	23	7
32	29	29	3	29	3
34	31	31	3	31	3
36	33	31	5	31	5
38	35	31	7	31	7
40	37	37	3	37	3
80	77	73	7	73	7
82	79	79	3	79	3
84	81	79	5	79	5
86	83	83	3	83	3
88	85	83	5	83	5
90	87	83	7	83	7
92	89	89	3	89	3
94	91	89	5	89	5
96	93	89	7	89	7
****98**	**95**	**89**	**9**	**79**	**19**
100	97	97	3	97	3
120	117	113	7	113	7
****122**	**119**	**113**	**9**	**109**	**13**
124	121	113	11	113	11
126	123	113	13	113	13
****128**	**125**	**113**	**15**	**109**	**19**

130	127	127	3	127	3
132	129	127	5	127	5
134	131	131	3	131	3
136	133	131	5	131	5
138	135	131	7	131	7
140	137	137	3	137	3
**500	497	491	9	487	13
502	499	499	3	499	3
504	501	499	5	499	5
506	503	503	3	503	3
508	505	503	5	503	5
510	507	503	7	503	7
1000	997	997	3	997	3
1002	999	997	5	997	5
1004	1001	997	7	997	7
**1006	1003	997	9	983	23
1008	1005	997	11	997	11
1010	1007	997	13	997	13
1012	1009	1009	3	1009	3
1014	1011	1009	5	1009	5
1016	1013	1013	3	1013	3
1018	1015	1013	5	1013	5
10002	9999	9973	29	9973	29
10004	10001	9973	31	9973	31
**10006	10003	9973	33	9923	83
**10008	10005	9973	35	9967	41
10010	10007	10007	3	10007	3
10012	10009	10009	3	10009	3
10014	10011	10009	5	10009	5
10016	10013	10009	7	10009	7
**10018	10015	10009	9	10007	11
10020	10017	10009	11	10009	11
2n - M	(2n -3) - M	W_{2n} - M	$T_{2n} = 2n - W_{2n}$	V_{2n} - M	U_{2n}
+1000	+997	+993	7	+993	7
**+1002	+999	+993	9	+931	71
+1004	+1001	+993	11	+993	11
+1006	+1003	+993	13	+993	13
**+1008	+1005	+993	15	+919	89

+1010	+1007	+993	17	+993	17
+1012	+1009	+993	19	+993	19
+1014	+1011	+1011	3	+1011	3
+1016	+1013	+1011	5	+1011	5
+1018	+1015	+1011	7	+1011	7
**+1020	+1017	+1011	9	+931	89
2n - R	(2n - 3) - R	W_{2n} - R	$T_{2n} = 2n - W_{2n}$	V_{2n} - R	U_{2n}
**+1000	+997	+979	21	+903	97
+1002	+999	+979	23	+979	23
**+1004	+1001	+979	25	+951	53
**+1006	+1003	+979	27	+903	103
+1008	+1005	+979	29	+979	29
+1010	+1007	+979	31	+979	31
**+1012	+1009	+979	33	+951	61
**+1014	+1011	+979	35	+781	233
+1016	+1013	+979	37	+979	37
**+1018	+1015	+979	39	+951	67
+1020	+1017	+1017	3	+1017	3
2n - G	(2n - 3) - G	W_{2n} - G	$T_{2n} = 2n - W_{2n}$	V_{2n} - G	U_{2n}
**+10000	+9997	+9631	369	+7443	2557
**+10002	+9999	+9631	371	+9259	743
+10004	+10001	+9631	373	+9631	373
**+10006	+10003	+9631	375	+8583	1423
**+10008	+10005	+9631	377	+6637	3371
+10010	+10007	+9631	379	+9631	379
**+10012	+10009	+9631	381	+8583	1429
+10014	+10011	+9631	383	+9631	383
**+10016	+10013	+9631	385	+9259	757
**+10018	+10015	+9631	387	+4491	5527
+10020	+10015	+9631	389	+9631	389
2n - S	(2n - 3) - S	W_{2n} - S	$T_{2n} = 2n - W_{2n}$	V_{2n} - S	U_{2n}
**+20000	+19997	+18031	1969	+17409	2591
**+20002	+19999	+18031	1971	+17409	2593
+20004	+20001	+18031	1973	+18031	1973
**+20006	+20003	+18031	1975	+16663	3343
**+20008	+20005	+18031	1977	+16941	3067
+20010	+20007	+18031	1979	+18031	1979
**+20012	+20009	+18031	1981	+5674	14341
**+20014	+20011	+18031	1983	+4101	15913
**+20016	+20013	+18031	1985	+3229	16787
+20018	+20015	+18031	1987	+18031	1987

****+20020**	**+20017**	**+18031**	**1989**	**+16941**	**3079**
2n - T	**(2n - 3) - T**	**W_{2n} - T**	**T_{2n} = 2n - W_{2n}**	**V_{2n} - T**	**U_{2n}**
****+40000**	**+39997**	**+29737**	**10263**	**+21567**	**18433**
****+40002**	**+39997**	**+29737**	**10265**	**+22273**	**17729**
+40004	+40001	+29737	10267	+29737	10267
****+40006**	**+40003**	**+29737**	**10269**	**+21567**	**18439**
+40008	+40005	+29737	10271	+29737	10271
+40010	+40007	+29737	10273	+29737	10273
****+40012**	**+40009**	**+29737**	**10275**	**+10401**	**29611**
****+40014**	**+40011**	**+29737**	**10277**	**-56003**	**96017**
****+40016**	**+40013**	**+29737**	**10279**	**+27057**	**12959**
****+40018**	**+40015**	**+29737**	**10281**	**+25947**	**14071**
****+40020**	**+40017**	**+29737**	**10283**	**+24493**	**15527**

11 Perspectives and Generalizations

11.1 Other Goldbach sequences (G'_{2n}) and (G''_{2n}) independent of (G_{2n}) may be studied using the increasing sequences of primes (W'_{2n}), (see 8.5) and (W''_{2n}) defined by :

For any integer n ≥ 3 , W''_{2n} = Sup(p ϵ $\mathcal{P}$: $p \leq$ f(n)), f being a function defined on the interval I = [3 ; +∞[and satisfying the following conditions:

* f is strictly increasing on the interval I,

* $lim_{x\to+\infty} f(x)$ = +∞ ; f(3) = 3 .

* $\forall\ x \ \epsilon\ I, f(x) \leq$ 2x − 3.

For example, one of the following functions defined on I can be selected.

a) f : x →ax + 3 - 3a ; (a ∈ ℝ : 0 < a ≤ 2).

b) g : x →[4$\sqrt{(3x)}$ − 9] ([x] being the integer part of the real number x) .

c) h : $x \rightarrow 6ln$ (x/3) + 3 .

11.2 Using this method, it would be interesting to study the Schnirelmann density [28] of certain primes such as 3 , 5 , 7, 11 ,..... ... in the sequence (U_{2n}) for n ∈ [K_N ; P_N] as a function of N .

11.3 It is possible to exceed the values shown in the table (2n = 10^{1000}) by optimizing this algorithm, using supercomputers and more efficients software as Maple .

11.4 Diophantine equations and conjectures of the same nature (Lagrange-Lemoine-Levy conjecture [9], [17], [19],[21],[22], [30]) can be treated using similar reasoning and algorithms.
1) To validate the Lagrange-Lemoine-Levy conjecture, we can study
the following sequences of primes (WL_{2n}), (VL_{2n}) and (UL_{2n}) defined by :

For any integer $n \geq 3$, WL_{2n}= Sup(p ϵ $\mathcal{P}$: $p \leq n - 1$),

a) If TL_{2n}= (2n + 1 - 2WL_{2n}) is a prime, then let :

$$VL_{2n} = WL_{2n} \quad \text{and} \quad UL_{2n} = TL_{2n}$$

b) If TL_{2n} is a composite number, then there exists an integer k , $(1 \leq k \leq n-3)$ such hat : $UL_{2(n-k)} + 2k$ is a prime ;

then let :

$$VL_{2n} = VL_{2(n-k)} \text{ and } UL_{2n} = UL_{2(n-k)} + 2k .$$

2) Using the same type of reasoning , a generalized Bezout-Goldbach conjecture of the following form can be validated :

a) Let K and Q be two odd integers, prime to each other : for any integer n such that : $(2n \geq 3(K+Q))$, there exist two primes U'''_{2n} and V'''_{2n} verifying : $K.U'''_{2n} + Q.V'''_{2n} = 2n$.

b) Let K and Q be two integers of different parity, prime to each other : for any integer n such that : $(2n \geq 3(K+Q))$, there are two primes U'''_{2n} and V'''_{2n} verifying : $K.U'''_{2n} + Q.V'''_{2n} = 2n + 1$.

12 Conclusion

12.1 An unique recurrent and explicit Goldbach sequence $(G_{2n}) = (U_{2n} \,;\, V_{2n})$, verifying : $(\forall\, n \in \mathbb{N} + 2,\ U_{2n}$ and V_{2n} are primes : $U_{2n} + V_{2n} = 2n)$, has been developed using an simple and efficient "local" algorithm.

12.2 Silva's [29] record is broken on a personal computer, and it is possible to reach values of the order of $2n = 10^{1000}$ with a reasonable computation time (less than three hours for the evaluation of ten terms U_{2n} and V_{2n}) .

12.3 For a given integer $n \geq 49$, the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and V_{2k} , $(1 \leq k < n - 1)$. we just need to know the primes p_l , V_{2r} such that :

(12.3.1) $p_l \leq 7(ln(2n))^{1.3}$ and $2n - 7(ln(2n))^{1.3} \leq V_{2r} \leq 2n$ **(on average).**

This property allows quick computing of U_{2n} and V_{2n} even for values of 2n of the order of 10^{1000} .

12.4 Therefore, the strong Euler-Goldbach and the Lagrange-Lemoine-Levy conjectures are true.

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